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Practical Methods for Elastic Collapse Analysis for Shell Structures

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Two-dimensional nonlinear numerical analysis of shells generally requires a substantial amount of computer time. A finite-difference based computer program was modified to handle grids with variable spacing. The resulting improvement in efficiency is demonstrated by use of practical examples. A study is presented also of different techniques for solution of the nonlinear equations. Such methods are evaluated by application to a number of example cases. The results apply to both nonlinear finite-element and finite-difference methods.

Nomenclature

a_{ij},b_{ij} = sides of rectangular region
f = function [see Eq. (1)]
h = $x_i - x_{i-1}$
k = $x_{i+1} - x_i$
w = lateral displacement
$\bar{w}$ = displacement at centroid of region
w_1,w_3 = see Fig. 2
x_i,y_j = values of coordinates at gridlines
$\bar{x}_i,\bar{y}_j$ = coordinates at centroid of region

I. Introduction

FOR analysis of the collapse of shells of general shape or loading and also for analysis of the post-buckling behavior of shells of revolution, it is necessary to perform a nonlinear analysis with two independent space coordinates. A two-dimensional nonlinear analysis is usually quite cumbersome and it is necessary for economy that the most practical methods be used. In Ref. 1, it was shown that use of a finite difference approach in combination with the energy principle provides a suitable basis for such analysis. A computer program STAGS was based on this method and is described in Refs. 2 and 3. Reference 2 gives some results for the collapse of cylindrical shells with cutouts. In Ref. 3, an extended version of the program is presented together with some numerical results for elliptical cylinders and cones.

For better economy in the analysis, a capability has now been provided for the use of variable grid spacing. The finite-difference expressions for a variable (rectangular) grid follow here.

II. Variable Mesh

The shell panel is covered with a system of mesh lines whose coordinates are given by x_i, $i = l, m$ and y_i, $i = 1, n$; where x and y are the curvilinear surface coordinates. Corresponding to each pair of values (i,j) $i = 1, m$; $j = 1, n$, a rectangular region $R_{i,j}$ (Fig. 1) is defined with sides of length

$$a_{i,j} \equiv \tfrac{1}{2}|x_{i+1} - x_{i-1}|;\ b_{i,j} \equiv \tfrac{1}{2}|y_{j+1} - y_{j-1}|$$

The regions $R_{i,j}$ (and lengths $a_{i,j},b_{i,j}$) are modified at boundaries of a shell by inclusion only of those portions which would be inside the panel. A double integral of a function

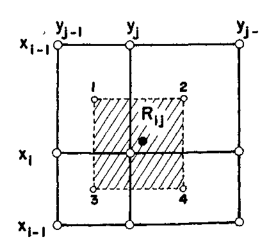

Fig. 1 Finite-difference mesh showing location of discrete variables.

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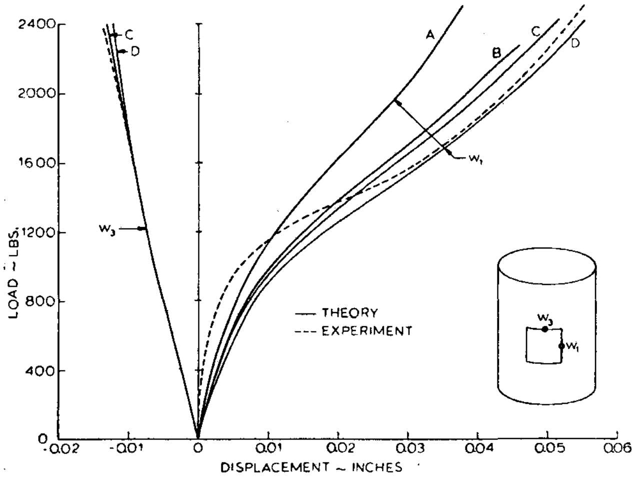

Fig. 2 Displacement vs load for cylinder with cutouts.

f over the region R of the panel may then be approximated by

$$\iint_R f dx dy = \sum_{i=1}^{m} \sum_{j=1}^{n} f_{i,j} a_{i,j} b_{i,j} \tag{1}$$

The discretization is completed when the integrand functions $f_{i,j}$ are evaluated at the centroids of the regions $R_{i,j}$ in terms of the neighboring displacement components. It should first be noted that the tangential displacements u and v have been located at corners of the regions $R_{i,j}$. Furthermore, the energy expressions for a shell include derivatives of u and v only up to the first order. Hence, even with arbitrary rectangular spacing, only central difference formulas for the u and v displacements are required. In contrast, the normal displacement w has been located at the mesh node points (x_i, y_j) and more general finite-difference formulas must be developed.

The coordinates of the centroid of a region $R_{i,j}$ are given by

$$\bar{x}_i = \tfrac{1}{4}(x_{i-1} + 2x_i + x_{i+1}); \; \bar{y}_j = \tfrac{1}{4}(y_{j-1} + 2y_j + y_{j+1}) \tag{2}$$

Variable spacing is first considered with respect to a single coordinate x only. With the help of a Taylor's expansion (or equivalently by the use of interpolation formulas), the difference formulas for w, $w_{,x}$ and $w_{,xx}$ at $\bar{x}_i$ may be established as

$$\begin{aligned}(w)_i \equiv w(\bar{x}_i) = {} & w_{i-1}/16[(h-k)(3k+h)/(h^2+hk)] + \\ & w_i/16[(h+3k)(3h+k)/(hk)] + \\ & w_{i+1}/16[(k-h)(3h+k)/(hk+k^2)]\end{aligned} \tag{3}$$

$$\begin{aligned}(w_{,x})_i \equiv w_{,x}(\bar{x}_i) = {} & -w_{i-1}/(2h) + \\ & w_i[1/(2h) - 1/(2k)] + w_{i+1}/(2k)\end{aligned} \tag{4}$$

$$\begin{aligned}(w_{,xx})_i \equiv w_{,xa}(\bar{x}_i) = {} & w_{i-1}2/[h(h+k)] - \\ & w_i 2/(hk) + w_{i+1}2/[k(h+k)]\end{aligned} \tag{5}$$

where

$$h = x_i - x_{i-1}; \quad k = x_{i+1} - x_i$$

The corresponding formulas for the y coordinate are obtained by appropriate substitutions and are denoted with

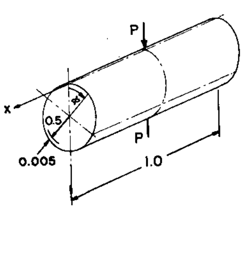

Fig. 3 Pinched cylinder.

Table 1 Displacement w_1 at $P = 845$ lb

Net	Minimum spacing	Factor	Maximum spacing	w_1
D (13 × 21)	0.12	1.5	0.60	0.00877
E (18 × 25)	0.12	1.2	0.60	0.00850
F (21 × 35)	0.12	1.2	0.30	0.00858
G (21 × 28)	0.08	1.2	0.60	0.00873

superscripts

$$(w)^j \equiv w(\bar{y}_j); \; (w_{,y})^j \equiv w_{,y}(\bar{y}_j); \; (w_{,yy})^j \equiv w_{,yy}(\bar{y}_j)$$

The required two-dimensional difference formulas are now obtained by combining the formulas for both coordinate directions

$$\bar{w}_{i,j} \equiv w(\bar{x}_i, \bar{y}_j) = [(w)_i]^j = [(w)^j]_i \tag{6a}$$

$$\bar{w}, \bar{x}\bar{x}_{i,j} \equiv w_{,xx}(\bar{x}_i, \bar{y}_j) = [(w_{,xx})_i]^j \tag{6b}$$

$$\bar{w}, \bar{y}\bar{y}_{i,j} \equiv w_{,yy}(\bar{x}_i, \bar{y}_j) = [(w_{,yy})^j]_i \tag{6c}$$

$$\bar{w}, \bar{x}\bar{y}_{i,j} \equiv w_{,xy}(\bar{x}_i, \bar{y}_i) = [(w_{,x})_{i,y}]^j \tag{6d}$$

In general, Eqs. (6) involve the 9 point "star" of neighboring values. However, it is easily seen that all of the formulas reduce to the standard central difference formulas when a uniform rectangular mesh is used.

The benefit derived from the use of a variable mesh spacing is first evaluated by re-examination of the cylinder problem for which test results were reported in Ref. 2. The cylinder has two diametrically opposite cutouts and a radius-to-thickness ratio of 400. It was reported in Ref. 2 that a reasonably accurate analysis for such cylinders would require excessive computer time. Numerical results for a finite difference net with 9 × 20 points presented in Ref. 2 are shown here in Fig. 2 (curve A). Because of improvements in the efficiency of the computer program, it is now possible to obtain much better results even with constant grid spacing. Curve B is obtained with such a net (16 × 20). A finite-difference mesh was designed also in which the minimum grid spacing is identical to that used for curve B, but which gradually increases away from the cutout until it is approximately doubled. The displacements corresponding to this analysis are practically identical to those obtained by use of a grid with constant spacing, but the computer time is reduced by about 40%.

Curve C was determined by use of a minimum grid spacing of 0.2 in. at the edge of the cutout. Moving away from the cutout the spacing increases by a factor of 1.2 from one mesh to the next until the maximum grid size of 0.6 in. is obtained. For curve D the minimum size is 0.12 in., the factor is 1.5 and the maximum size is again 0.6 in. It appears that the results obtained by use of the latter mesh are in very good agreement with the experimental results.

The computer time corresponding to the determination of one of these curves is approximately 0.5 hr (Univac 1108). For analyses with even finer mesh sizes, therefore, the analysis was restricted to loads below 845 lb. The results in Table 1 indicate that additional refinement of the mesh would not substantially change the results shown in curve D.

The case of a pinched cylinder, Fig. 3, was also studied. Lateral displacements computed from a linear analysis are shown versus the circumferential coordinate in Fig. 4a and vs the axial coordinate in Fig. 4b. The curves are for a converged solution, corresponding to a variable spacing grid with 17 points in the axial and 26 points in the circumferential directions. These results are in good agreement with results for the same case shown in Ref. 4. Discrete values of the solution are given for the two coarser nets (A and B) which are shown in Fig. 5. It can be seen that the use of the net with variable spacing, Grid A, leads to results which are at least of the same quality as those obtained with the uniform

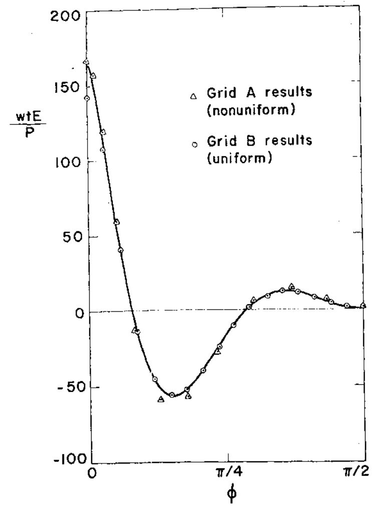

Fig. 4a Lateral displacements (vs circumferential coordinate) for pinched cylinder.

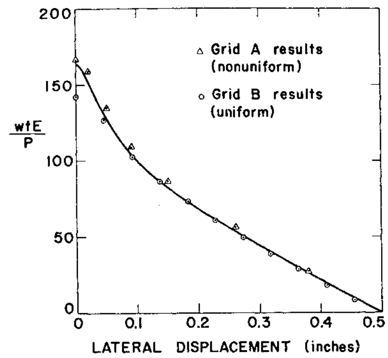

Fig. 4b Lateral displacements (vs axial coordinate) for pinched cylinder.

net, Grid B. The computer time corresponding to the analysis with Grid B is approximately 5 times the time for analysis with Grid A.

III. Solution of Nonlinear Equations

The discretization of the system by use of finite-difference approximation leads to a set of nonlinear algebraic equations. The Newton-Raphson method is a very efficient and widely used method for solution of such equation systems.

In solution of a shell collapse problem equilibrium configurations are determined which correspond to gradually increasing values of the applied load. If a maximum occurs in the load displacement curve, this maximum represents the collapse load. As the Newton-Raphson method requires relatively good estimates for the unknowns, such estimates are generally obtained through extrapolation from values of previous load steps. If the case under consideration is such that substantial redistribution of stresses takes place before

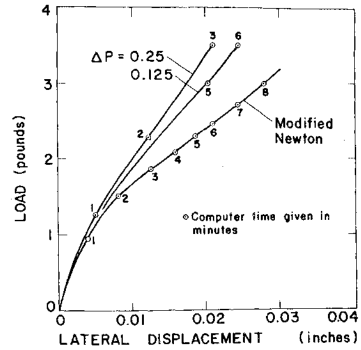

Fig. 6 Pinched cylinder incremental method.

collapse occurs, the step size must be relatively small as otherwise the procedure will fail to converge. Additional restrictions may have to be applied to the step size for inelastic analysis due to path-dependence of the constitutive equations.

In Ref. 2, it was found that the modified Newton-Raphson methods offers considerable advantage over the regular method. Convergence, in this case, takes a few more iterations but the time required for each iteration is much less than in the regular method.

Other methods have been used for solution of the nonlinear equation system. In the method with successive substitutions, nonlinear terms are added to the load vector. This method suffers from convergence difficulties, Ref. 5, and is probably not suitable for collapse analysis. Two other methods were discussed in Ref. 6. One of these, the incremental analysis, has been quite popular particularly in connection with finite-element analysis. It is assumed in application of this method that for a fixed load level, $p = p_0$, the displacement pattern corresponding to equilibrium, $u = u_0$, is known. The nonlinear equilibrium equations are expressed for $p = p_0 + \Delta p$ and $u = u_0 + \Delta u$. In these equations all, terms containing only p_0,u_0 can be subtracted out due to equilibrium of the configuration p_0,u_0. For sufficiently small values of Δp, Δu will be small and only first order terms need be retained. With $p_0 = 0$, $u_0 = 0$ as the original known solution one can with small steps compute u for successively increasing values of p. However, as each of the p_0,u_0 configurations only represents an approximate solution, errors will be accumulated during the process and it may be necessary to use an extremely small step size to achieve reasonable accuracy in the results.

An improvement in the incremental method is proposed in Ref. 6. At each load increment as a solution $u = u_0 + \Delta u$ is obtained, this solution is substituted into the nonlinear equations and the unbalance in each equation is determined and subsequently added to the right-hand side for the next load step. Although errors will still accumulate when this method is used, this accumulation will be less severe. It is interesting to notice that this improved incremental method

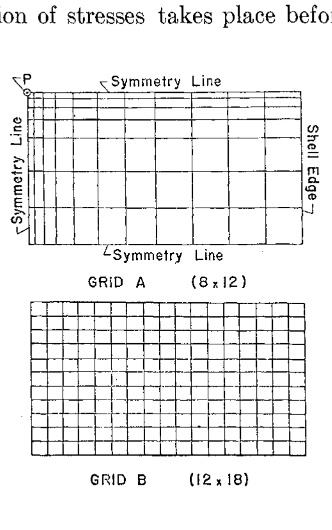

Fig. 5 Finite-difference grids.

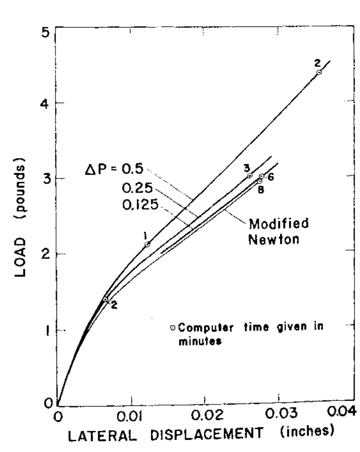

Fig. 7 Pinched cylinder one-step Newton method.

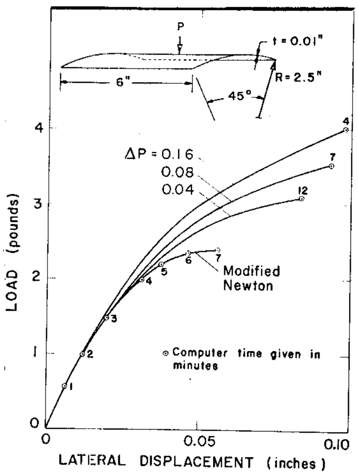

Fig. 8 Venetian blind, incremental method.

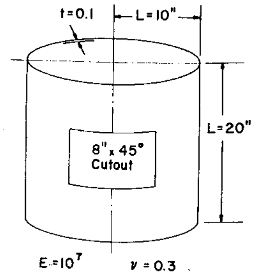

Fig. 10 Cylinder with cutouts.

leads to exactly the same analysis as does a Newton-Raphson method with only one iteration at each load step. The method will be referred to here as the one-step Newton method.

The computer program STAGS was based on the modified Newton-Raphson method but it can now, after minor modification, also be used for incremental analysis of either of the types discussed here. It is applied here in a study of the efficiency of the three different techniques for nonlinear analysis.

Three different shell structures were analyzed. The results for the pinched cylinder (Fig. 3) with Grid A (Fig. 5) are shown in Figs. 6 and 7. Figure 6 shows the results obtained by use of the modified Newton method, representing an accurate solution of the nonlinear equations, together with results from the incremental method. The load is plotted versus the lateral displacement at the point where the load is applied. By use of the modified Newton method, computation of the load displacement curve up to 3 lb requires 8 min (Univac 1108) computer time. It appears that using the incremental method we must accept about 20% error in order that the computer time not be more than 8 min. In Fig. 7, the one-step Newton method is evaluated for the same case. If the step size is chosen so that the computer time (up to 3.0 lb load) is about 8 min, the error in the computed displacement is less than 2%. If a 10% accuracy is accepted, the same results can be obtained in less than 3 min.

The other two shell structures for which comparisons were made differs from the pinched cylinders inasmuch as the critical failure mode for these is collapse. The results for a cylindrical panel subject to bending by a point force are shown in Figs. 8 and 9. The longitudinal edges of the panel were free and the curved edges were simply supported. The finite-difference grid for a quarter of the shell has 10 axial and 8 circumferential stations. With the modified Newton method 8 min is sufficient for the establishment of a collapse load, 2.48 lb. The incremental method Fig. 8 is very inefficient in this case. With the one-step Newton method and a rather small step size, the solution is quite close to the converged solution (modified Newton) up to the point of maximum. However, the curve fails to bend over at that point and no collapse is indicated.

The third case is shown in Fig. 10. A cylinder with two diametrically opposite rectangular cutouts is loaded through uniform axial shortening. The lateral outward displacement midlength of the edge of the cutout is plotted vs the axial shortening in Figs. 11 and 12. With the modified Newton method, a critical axial load of 134,000 lb is determined at the price of about 10 min of computer time. From Fig. 11, it seems that the incremental method works better than in any of the other two cases. This may be because there are two effects here which compete with one another. The geometry of the shell deteriorates due to bending of the shell wall at the cutout edge. However, collapse of the shell is delayed because the stresses are gradually redistributed with increasing load. The collapse load for the shell is not well defined here. At some load level the determinant is not positive definite, indicating unstable equilibrium, or a wild solution is obtained. The critical loads indicated in the figure represent the last load level for which a reasonable solution was obtained.

The results obtained by use of the one-step Newton method are shown in Fig. 12. They are similar to the results from the incremental analysis and the critical load values are determined in the same way.

Comparing the three different methods for solution of the nonlinear system we must, of course, keep in mind that if the

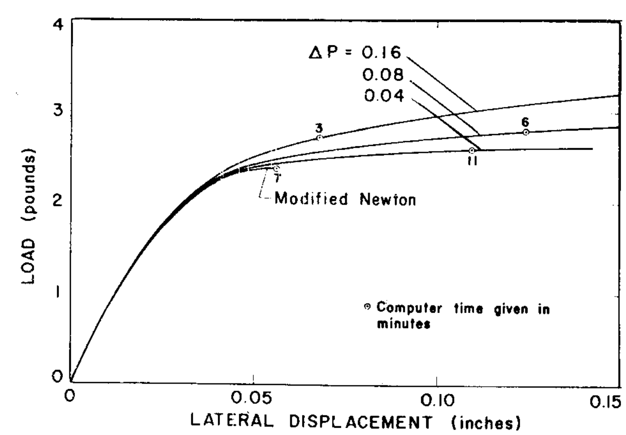

Fig. 9 Venetian blind, one-step Newton method.

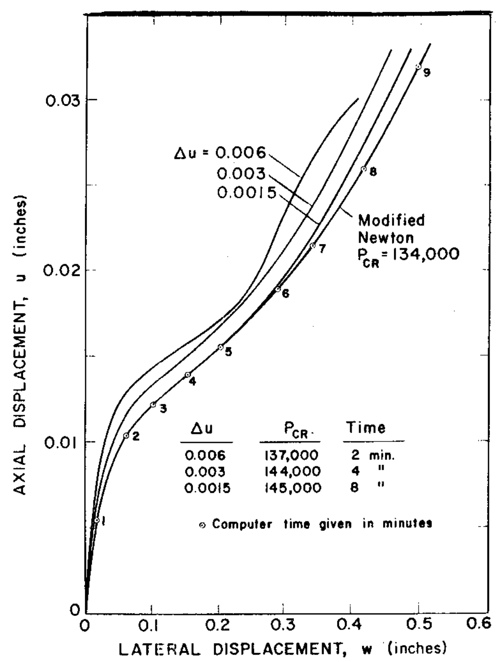

Fig. 11 Cylinder with cutout, incremental method.

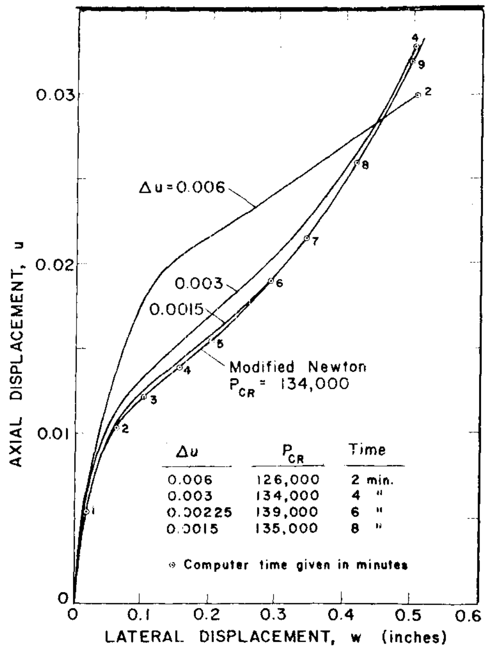

Fig. 12 Cylinder with cutout, one-step Newton method.

modified Newton is used a converged solution is automatically obtained. With any of the other two methods we must in general run a series of analyses with varying step size in order to find one that gives sufficiently good results. Also all these cases have a relatively small bandwidth in the coefficient matrix. An increase in this bandwidth would more heavily penalize the incremental and the one-step Newton methods because the run time for matrix decomposition increases more rapidly than the time required for iterations of the modified Newton method. Thus, it appears that the modified Newton method almost always would be preferred. The incremental method is not appreciably faster than the one-step Newton method and only the latter method can therefore be considered as a viable alternative. If results from similar cases are available so that a suitable step size can be determined in advance, we may save computer time for some cases by use of the one-step Newton method. The relatively successful results with this method may indicate also that the modified Newton method may be improved by more flexibility in the chosen strategy.

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